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SCHRÖDINGER OPERATORS WITH MAGNETIC FIELDS AND MINIMAL ACTION FUNCTIONALS

BY

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ABSTRACT

We consider a convex superlinear Lagrangian L on a closed connected manifold M such that its associated Hamiltonian H satisfies controllable growth conditions. For this class of Lagrangians we define H-harmonic functions and the harmonic value h and we compare it with Mañé's critical value c. We show, using elliptic regularity of quasilinear elliptic equations, that $h \leq c$ and the equality holds iff there exists a unique (up to constants) smooth weak KAM solution which is H-harmonic.

Fix a Riemannian metric on M with volume one and consider a real C^∞ 1-form θ and a smooth function $V: M \to \mathbb{R}$. Let L be the convex and superlinear Lagrangian given by

$$L(x, v) := \frac{1}{2}|v|_x^2 + \theta_x(v) - V(x).$$

This is a special but important class of Lagrangians. We consider the Schrödinger operator $\mathbb{H}_{(\theta,V)}$ associated with L and we let λ_0 be its first eigenvalue. We show that $\lambda_0 \leq h$ with equality only if $h = c$. When $h = c$ this common value is an eigenvalue of $\mathbb{H}_{(\theta,V)}$, but not necessarily the smallest one. Using these ideas we define a norm $|\cdot|_{Schr}$ in $H^1(M, \mathbb{R})$ that we call the Schrödinger norm and we compare it with the L^2-norm $|\cdot|_{L^2}$ and with the stable norm $|\cdot|_s$. We show that for any cohomology class $[\omega] \in H^1(M, \mathbb{R})$ we have

$$|[\omega]|_{Schr} \leq |[\omega]|_{L^2} \leq |[\omega]|_s\,.$$

Any of the inequalities is an equality if and only if the unique harmonic representative in $[\omega]$ has constant Riemannian norm. We derive various corollaries from these results.

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1. Introduction

Let M be a closed connected smooth manifold and let $L: TM \to \mathbb{R}$ be a smooth convex superlinear Lagrangian. This means that L restricted to each T_xM has positive definite Hessian and that for some Riemannian metric we have that

$$\lim_{|v|\to\infty} \frac{L(x,v)}{|v|} = \infty,$$

uniformly on $x \in M$. Let H: $T^*M \to \mathbb{R}$ be the Hamiltonian associated to L and let $\mathcal{L}$: $TM \to T^*M$ be the Legendre transform $(x, v) \mapsto \partial L/\partial v(x, v)$. Since M is compact, the extremals of L give rise to a complete flow ϕ_t: $TM \to TM$ called the Euler-Lagrange flow of the Lagrangian. Using the Legendre transform we can push forward ϕ_t to obtain another flow ϕ_t^* which is the Hamiltonian flow of H with respect to the canonical symplectic structure of T^*M. The energy of L is the function E: $TM \to \mathbb{R}$ given by

$$E(x,v) = \frac{\partial L}{\partial v}(x,v)(v) - L(x,v).$$

The energy E is a first integral of the Euler-Lagrange flow ϕ_t and $H o \mathcal{L} = E$.

Recall that the action of the Lagrangian L on an absolutely continuous curve γ: $[a, b] \to M$ is defined by

$$A_L(\gamma) = \int_a^b L(\gamma(t), \dot\gamma(t))\, dt.$$

Given two points, x_1 and x_2 in M and $T > 0$ denote by $\mathcal{C}_T(x_1, x_2)$ the set of absolutely continuous curves γ: $[0, T] \to M$, with $\gamma(0) = x_1$ and $\gamma(T) = x_2$. For each $k \in \mathbb{R}$ we define the *action potential* Φ_k: $M \times M \to \mathbb{R}$ by

$$\Phi_k(x_1, x_2) = \inf\{A_{L+k}(\gamma) : \ \gamma \in \cup_{T>0}\mathcal{C}_T(x_1, x_2)\}.$$

The *critical value of* L, which was introduced by Mañé in [18], is the real number $c := c(L)$ defined as the infimum of $k \in \mathbb{R}$ such that for some $x \in M$ (and hence for all x), $\Phi_k(x, x) > -\infty$. Since L is convex and superlinear and M is compact such a number exists and its importance can hardly be overestimated. The critical value singles out the energy level in which relevant globally minimizing objects (orbits or measures) live [9, 18, 5]. The study of these globally minimizing objects has a long history that goes back to M. Morse and G. A. Hedlund. Recent work on this subject has been done by V. Bangert [1, 2], M. J. Dias Carneiro [9], A. Fathi [10, 11], R. Mañé [18, 19] and J. Mather [21, 22].

The critical value can be characterized in a variety of ways [18, 5, 7, 8]. Each of these characterizations gives a new insight into the geometry and the dynamics of the Lagrangian L. Let us explain first the relation of the critical value with Mather's theory of minimizing measures [21]. Let $\mathcal{M}(L)$ be the set of probabilities on the Borel σ-algebra of TM that have compact support and are invariant under the Euler-Lagrange flow ϕ_t. Let $H^1(M,\mathbb{R})$ be the first de Rham cohomology group of M and let α: $H^1(M,\mathbb{R}) \to \mathbb{R}$ be Mather's minimal action functional, which is given by (cf. [21])

$$\alpha([\omega]) = -\min\left\{\int (L-\omega)\,d\mu : \ \mu \in \mathcal{M}(L)\right\}. \tag{1}$$

Mañé [18, 5] showed that

$$c(L) = -\min\left\{\int L\,d\mu : \ \mu \in \mathcal{M}(L)\right\}, \tag{2}$$

and therefore combining (1) and (2) we obtain the remarkable relation

$$c(L-\omega) = \alpha([\omega]), \tag{3}$$

for any closed 1-form ω whose cohomology class is $[\omega]$.

On the other hand, the critical value $c(L)$ can also be recovered purely out of the Hamiltonian as the following result obtained in [7] (and also independently by Fathi) shows:

$$c(L) = \inf_{u\in C^\infty(M,\mathbb{R})} \max_{x\in M} H(x, d_x u), \tag{4}$$

where H is the Hamiltonian associated with L. This invites us to regard the theory of minimizing measures as the variational theory of the functional

$$\mathbb{M}\colon C^\infty(M,\mathbb{R}) \to \mathbb{R},$$

given by

$$\mathbb{M}(u) := \max_{x\in M} H(x, d_x u). \tag{5}$$

We do not know if the infimum of $\mathbb{M}$ is always achieved at a smooth function, even though we suspect that generically in the sense of Mañé [19] this is the case. However, if one searches for minimizers of the functional $\mathbb{M}$ within the space of Lipschitz functions then one can easily show that for any $y \in M$, the function $u(x) := \Phi_c(y,x)$ is a minimizer (for any $k \geq c$, the action potential Φ_k is a Lipschitz function that satisfies a triangle inequality [18, 5]). In fact, a lot more can be said since one can find minimizers with much better properties, as we now explain. This is the content of Fathi's weak KAM theorem [10]. Before explaining it, we need a few definitions.

We say that an absolutely continuous curve $\gamma\colon [a,b] \to M$ is *semistatic* if

$$A_{L+c}\left(\gamma|_{[s,t]}\right) = \Phi_c(\gamma(s), \gamma(t))$$

for all $a \leq s \leq t \leq b$. Semistatic curves are solutions of the Euler-Lagrange equation because of their minimizing properties. Also, it is not hard to check that semistatic curves have energy precisely c [18, 5].

Given a continuous function $u\colon M \to \mathbb{R}$, we shall write $u \prec L + c$ whenever $u(x) - u(y) \leq \Phi_c(y, x)$ for all $x, y \in M$. It is easy to check that if $u \prec L + c$ then u must be Lipschitz (cf. [14]).

We shall say that a continuous function $u_+\colon M \to \mathbb{R}$ is a *positive weak KAM solution* if u_+ satisfies the following two conditions:

1. $u_+ \prec L + c$;
2. for all $x \in M$, there exists a semistatic curve $\gamma_+^x\colon [0, \infty) \to M$ such that

$$u_+(\gamma_+^x(t)) - u_+(x) = \Phi_c(x, \gamma_+^x(t)), \text{ for all } t \geq 0.$$

Similarly, we shall say that a continuous function $u_-\colon M \to \mathbb{R}$ is a *negative weak KAM solution* if u_- satisfies the following two conditions:

1. $u_- \prec L + c$;
2. for all $x \in M$, there exists a semistatic curve $\gamma_-^x\colon (-\infty, 0] \to M$ such that

$$u_-(x) - u_-(\gamma_-^x(-t)) = \Phi_c(\gamma_-^x(-t), x), \text{ for all } t \geq 0.$$

Fathi's weak KAM theorem asserts that positive and negative weak KAM solutions always exist. His proof is based on applying the Banach fixed point theorem to the semigroup of operators induced by the Lax-Oleinik semigroup on the space of continuous functions on M divided by the constant functions (see [7] for a different proof and a non-compact version of this result).

By property 1, weak KAM solutions are Lipschitz and by Rademacher's theorem, they are differentiable almost everywhere. At any point x of differentiability of a weak KAM solution u, conditions 1 and 2 imply that

$$H(x, d_x u) = c.$$

Hence we see that weak KAM solutions are minimizers of the functional $\mathbb{M}$ within the class of Lipschitz functions.

Fathi's approach gives more than the existence of positive and negative weak KAM solutions. We refer to [10, 11, 12, 13, 14] for more information. See also

G. Contreras' paper [4] for a complete description of the relationship between action potentials and weak KAM solutions.

From now on we shall assume that H satisfies the *controllable growth conditions.* This means that there are positive constants k, m_1 and m_2 and a finite covering of M by local charts (Ω_i, φ_i) such that on each chart we have

$$\left|\frac{\partial H}{\partial p}(x,p)\right|^2 + \left|\frac{\partial^2 H}{\partial p \partial x}(x,p)\right|^2 \leq k\,(1+|p|_x^2),$$
$$m_1|\xi|_x^2 \leq \frac{\partial^2 H}{\partial p^2}(x,p)(\xi) \leq m_2|\xi|_x^2,$$

for all $(x,p) \in T^*\Omega_i$ and $\xi \in T_x^*\Omega_i$. As we explain in Section 2, this is not a restriction at all if one is looking at the variational theory of the functional $\mathbb{M}$. On the other hand, this controllable growth on the fibres will allow us to use the elliptic regularity results associated to quasilinear elliptic equations.

Suppose now that instead of considering the functional $\mathbb{M}$, we consider the functional

$$\mathbb{I}\colon C^\infty(M,\mathbb{R}) \to \mathbb{R},$$

given by

$$\mathbb{I}(u) := \int_M H(x, d_x u)\, dx, \tag{6}$$

where dx is the normalized Riemannian measure of some Riemannian metric. Let us define the *harmonic value* $h := h(L)$ of the Lagrangian L as

$$h(L) = \inf_{u \in C^\infty(M,\mathbb{R})} \mathbb{I}(u). \tag{7}$$

The variational theory of the functional $\mathbb{I}$ has a different flavour to that of the functional $\mathbb{M}$. Integration smooths out the minimizers. The Euler-Lagrange equation of $\mathbb{I}$ is a quasilinear elliptic equation and, as we explain in Section 2, the infimum of $\mathbb{I}$ is achieved at a C^∞ function and the minimizer is unique up to constants. We call such a minimizer an *H-harmonic* function, because when L is given by a Riemannian metric, the minimizers of $\mathbb{I}$ are just the harmonic functions, i.e., the constants.

It follows right away from (4), (6) and (7) that

$$h \leq c.$$

In Section 3 we show

THEOREM A: *The harmonic value and the critical value coincide if and only if the set of weak KAM solutions coincides with the set of H-harmonic functions. In other words, the harmonic value and the critical value coincide if and only if there exists a unique smooth weak KAM solution, up to constants, which is H-harmonic.*

In Section 3 we shall show the following corollary of Theorem A:

COROLLARY 1: *Given a Lagrangian L there exists a unique $\psi \in C^\infty(M,\mathbb{R})$ (up to constants) for which $h(L+\psi) = c(L+\psi)$.*

In other words, the corollary says that there is a unique way of modifying a Lagrangian with a potential in such a way that the set of weak KAM solutions coincides with the set of H-harmonic functions.

We now consider a special, but important class of Lagrangians. Fix a Riemannian metric on M with volume one and consider a real C^∞ 1-form θ and a smooth function $V\colon M \to \mathbb{R}$. Let L be the convex and superlinear Lagrangian given by

$$L(x,v) := \frac{1}{2}|v|_x^2 + \theta_x(v) - V(x).$$

Its associated Hamiltonian is

$$H(x,p) = \frac{1}{2}|p-\theta_x|_x^2 + V(x).$$

As we explain in Section 4, the Dirac quantization rule gives rise to the Schrödinger operator

$$\mathbb{H}_{(\theta,V)}u = \frac{1}{2}\Delta u - i\langle du,\theta\rangle + \left(\frac{i}{2}d^*\theta + \frac{1}{2}|\theta|^2 + V\right)u,$$

where d^* is the formal adjoint of the exterior derivative, u is a smooth function with values in $\mathbb{C}$ and $\langle\ ,\ \rangle$ is the Hermitian inner product in T^*M induced by the Riemannian metric. The operator $\mathbb{H}_{(\theta,V)}$ is known to be essentially self-adjoint and it has a real discrete spectrum

$$\lambda_0 \le \lambda_1 \le \lambda_2 \le \cdots \to \infty.$$

Other properties that we shall need from this operator are summarized in Section 4 and are taken from [26]. In the same section we show:

THEOREM B: *We have*

$$\lambda_0 \le h \le c.$$

If $\lambda_0 = h$, then $h = c$. If $h = c$, then this common value is an eigenvalue of $\mathbb{H}_{(\theta,V)}$.

Let $H^1(M,\mathbb{Z}) \subset H^1(M,\mathbb{R})$ be the lattice of integral cohomology classes. We shall see in Section 4 that the spectrum of $\mathbb{H}_{(\theta,V)}$ is the same as the spectrum of $\mathbb{H}_{(\theta-\omega,V)}$ for any closed 1-from ω such that $[\omega]/2\pi \in H^1(M,\mathbb{Z})$. Hence, Theorem B implies right away that

$$\lambda_0 \leq \min_{[\omega]/2\pi\in H^1(M,\mathbb{Z})} h(L-\omega) \leq \min_{[\omega]/2\pi\in H^1(M,\mathbb{Z})} c(L-\omega).$$

In general we cannot claim that if $h = c$, then this common value is the *smallest* eigenvalue (see examples in Section 4). However, some information can be obtained if we specialize further by considering the case when V vanishes identically.

Let ε be a small parameter. Since the smallest eigenvalue of $\mathbb{H}_{(0,0)} = \Delta/2$ is simple, then $\lambda_0(\varepsilon)$, the smallest eigenvalue of $\mathbb{H}_{(\varepsilon\theta,0)}$, is also simple and the map $\varepsilon \mapsto \lambda_0(\varepsilon)$ is real analytic by standard perturbation theory [17]. We use the expansions obtained by I. Schigekawa in [26] to show that for ε small we have

$$\lambda_0(\varepsilon) = \varepsilon^2 h + \text{higher order terms,} \tag{8}$$

and that

$$\lambda_0(\varepsilon) = \varepsilon^2 h,$$

for all ε sufficiently small iff $h = c$.

Finally, we consider the case in which θ vanishes identically, that is, L is just given by the Riemannian metric. In this case it is known that Mather's α function and the stable norm in cohomology ([23]) are related by

$$|[\omega]|_s = \sqrt{2\alpha([\omega])}.$$

Let ω be a closed 1-form. By considering the Lagrangian $L-\omega$, whose Euler-Lagrange flow coincides with the geodesic flow, we have functions on $H^1(M,\mathbb{R})$ given by

$$[\omega] \mapsto h(L-\omega),$$
$$[\omega] \mapsto \lambda_0(\omega) = \text{smallest eigenvalue of } \mathbb{H}_{(\omega,0)}.$$

We shall see that $\sqrt{2h(L-\omega)}$ is nothing but the L^2-norm of the unique harmonic 1-form in the class $[\omega]$. We denote it by $|[\omega]|_{L^2}$. By standard perturbation theory and (8) the function

$$[\omega] \mapsto \lambda_0(\omega)$$

is real analytic and strictly convex in a neighborhood of the origin. Hence for any σ small and positive, $\lambda_0^{-1}(\sigma)$ is a convex hypersurface in $H^1(M,\mathbb{R})$ symmetric about the origin. This hypersurface can be used to define a third norm in $H^1(M,\mathbb{R})$. We call it the *Schrödinger norm* and it is given by

$$||[\omega]||_{Schr,\sigma} = \frac{\sqrt{2\sigma}}{r},$$

where r is the unique positive real number such that $r[\omega] \in \lambda_0^{-1}(\sigma)$.

In Section 5 we show

THEOREM C: *For any cohomology class $[\omega] \in H^1(M,\mathbb{R})$ we have*

$$||[\omega]||_{Schr,\sigma} \leq ||[\omega]||_{L^2} \leq ||[\omega]||_s .$$

Any of the inequalities is an equality if and only if the unique harmonic representative in $[\omega]$ has constant Riemannian norm. Moreover

$$\lim_{\sigma\to 0} ||[\omega]||_{Schr,\sigma} = ||[\omega]||_{L^2} .$$

In [23, Equation (4.5)] D. Massart showed for Riemann surfaces that $||[\omega]||_{L^2} \leq ||[\omega]||_s$ using Strebel forms.

In Section 5 we show the following consequence of Theorem C:

COROLLARY 2: *Suppose that the first Betti number of M coincides with the dimension of M. Then the following are equivalent:*

1. *The three norms coincide for some σ.*
2. *$|\cdot|_{Schr,\sigma}$ is independent of σ.*
3. *M is a flat torus.*

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2. H-harmonic functions and the harmonic value

Let $L: TM \to \mathbb{R}$ be a C^∞ convex superlinear Lagrangian on a closed connected manifold M and let H be its associated Hamiltonian. We shall say that H satisfies the *controllable growth conditions* if there are positive constants k, m_1 and m_2 and a finite covering of M by local charts (Ω_i, φ_i) such that on each chart we have

$$\left|\frac{\partial H}{\partial p}(x,p)\right|^2 + \left|\frac{\partial^2 H}{\partial p \partial x}(x,p)\right|^2 \leq k\,(1+|p|_x^2), \tag{9}$$

$$m_1|\xi|_x^2 \leq \frac{\partial^2 H}{\partial p^2}(x,p)(\xi) \leq m_2|\xi|_x^2, \tag{10}$$

for all $(x,p) \in T^*\Omega_i$ and $\xi \in T_x^*\Omega_i$.

Consider a convex superlinear Lagrangian L and $k > c(L)$. We can always modify L outside the compact set given by those (x,v) satisfying $E_L(x,v) \leq k+1$ to obtain a Lagrangian L_0 whose associated Hamiltonian satisfies the controllable growth conditions. For example, it suffices to make L_0 a function that depends quadratically on the velocities for all (x,v) with $|v|_x$ sufficiently large. Since L and L_0 agree on the set given by those (x,v) satisfying $E_L(x,v) \leq k+1$ and $k > c(L)$, it is quite simple to check using (4) that $c(L_0) = c(L)$. Since all the globally minimizing objects that we described in the introduction live in the energy level $E_L^{-1}(c(L))$, this means that assuming that the Hamiltonian of L satisfies the controllable growth conditions is not a restriction at all if one is looking at the variational theory of the functional $\mathbb{M}$. On the other hand, this controllable growth on the fibres allows us to use the elliptic regularity results associated to quasilinear elliptic equations that we now recall.

Suppose now that M is a closed connected manifold and let L be a convex superlinear Lagrangian whose Hamiltonian satisfies the controllable growth conditions. Endow M with the Riemannian metric and let dx be its normalized Riemannian measure. Let $H^{1,2}(M)$ be the Sobolev space of functions u on M such that $u \in L^2(M,dx)$ and such that $x \mapsto |d_x u|$ in the sense of distributions is also in $L^2(M,dx)$.

Let H be the Hamiltonian associated with L and let us consider the functional $\mathbb{I}\colon H^{1,2}(M) \to \mathbb{R}$ given by

$$\mathbb{I}(u) := \int_M H(x, d_x u)\, dx.$$

By considering an expansion of order two of the function $t \mapsto H(x,tp)$ and using (10), it is immediate to check that there are positive constants a_1, a_2 and a_3 such that

$$a_1|p|_x^2 - a_2 \leq H(x,p) \leq a_3|p|_x^2 + a_2,$$

for all $(x,p) \in T^*M$. The direct methods in the calculus of variations show the following (see, for example, [28, Theorem 11.6] or [15, 24]).

THEOREM 2.1: *The functional* $\mathbb{I}$ *has a minimum in* $H^{1,2}(M)$.

Let u_0 be a minimum of $\mathbb{I}$ and let $v \in C^\infty(M, \mathbb{R})$. For $t \in (-1, 1)$ we can consider the differential quotient at zero of the function $I(t) = I(u_0 + tv)$ given by

$$\frac{I(t) - I(0)}{t} = \int_M dx \int_0^1 D_p H(x, d_x u_0 + s t d_x v)(d_x v)\, ds,$$

where $D_p H$ is the fibre derivative of H. Since H satisfies the controllable growth conditions, the uniform summability theorem of Lebesgue ensures that we can pass to the limit for $t \to 0$ getting that u_0 satisfies

$$\int_M D_p H(x, d_x u_0)(d_x v)\, dx = 0$$

for all $v \in C^\infty(M, \mathbb{R})$. In other words, u_0 is a *weak solution* of the quasilinear elliptic equation

$$\operatorname{div} D_p H(x, d_x u) = 0, \tag{11}$$

where we regard $x \mapsto D_p H(x, d_x u)$ as a vector field on M using the duality $(T_x^* M)^* = T_x M$ and div is the divergence operator of the Riemannian metric. In local coordinates we can write equation (11) as

$$\frac{\partial}{\partial x^i} \left(\sqrt{g}\, g^{ij} \frac{\partial H}{\partial p^j}(x, du) \right) = 0. \tag{12}$$

Let

$$A^i(x, p) = \sqrt{g}\, g^{ij} \frac{\partial H}{\partial p^j}(x, p);$$

hence we can rewrite (12) as

$$\frac{\partial}{\partial x^i} A^i(x, du) = 0. \tag{13}$$

Our assumptions on H imply that there exist positive constants m_1, m_2 and k_1 and k_2 such that

$$\sum_i |A^i|^2 + \sum_{i,j} \left| \frac{\partial A^i}{\partial x^j} \right|^2 \leq k_1 (1 + |p|^2), \tag{14}$$

$$\sum_{i,j} \left| \frac{\partial A^i}{\partial p^j} \right|^2 \leq k_2, \tag{15}$$

$$m_1 |p|^2 \leq \frac{\partial A^i}{\partial p^j} p^i p^j \leq m_2 |p|^2. \tag{16}$$

THEOREM 2.2 (Di Giorgi–Moser–Morrey): *Let $u \in H^{1,2}(\Omega)$ be a weak solution of* (13) *where A^i satisfy* (14), (15) *and* (16). *Then, locally the first derivatives of u satisfy a Hölder condition depending only on $|du|_2$, m_1, m_2, k_1 and k_2. Furthermore, u is $C^{2+\alpha}$. If A^i is C^∞ then u is C^∞; if A^i is analytic then u is analytic.*

The results in the theorem stated above can be found, for example, in [16, 24, 28].

This theorem implies that any minimum of the functional $\mathbb{I}$ on $H^{1,2}(M)$ must be a C^∞ function.

Now that we know that the minimizers of $\mathbb{I}$ are C^∞, it is quite easy to check using the convexity of H that in fact there is only one minimizer modulo constants. Indeed, let u and v be two minimizers of $\mathbb{I}$ and suppose that for some $x \in M$, $d_x(u-v) \neq 0$. Given $t \in \mathbb{R}$, let $I(t)$ be the smooth function given by $I(t) := I(tu + (1-t)v)$. Note that

$$I''(t) = \int_M D_{pp}H(x, td_xu + (1-t)d_xv)(d_x(u-v), d_x(u-v))\, dx.$$

Since H is strictly convex we have that $I''(t) > 0$ for all t. But $t = 0$ and $t = 1$ are absolute minimums of I. This contradiction shows that in fact $d_x(u-v) \equiv 0$, or equivalently u and v only differ in a constant.

We summarize the results obtained so far in the following theorem.

THEOREM 2.3: *Let M be a closed connected manifold. Let L: $TM \to \mathbb{R}$ be a C^∞ convex superlinear Lagrangian whose associated Hamiltonan Hsatisfies the controllable growth conditions. Let $\mathbb{I}$: $H^{1,2}(M) \to \mathbb{R}$ be the functional given by*

$$\mathbb{I}(u) := \int_M H(x, d_xu)\, dx.$$

Then the infimum of $\mathbb{I}$ is achieved at a C^∞ function and the minimizer is unique up to constants. Moreover the following three conditions are equivalent:

1. *u is a minimizer of $\mathbb{I}$;*
2. *u is a weak solution of* $\operatorname{div} D_pH(x, d_xu) = 0$;
3. *u is a strong solution of* $\operatorname{div} D_pH(x, d_xu) = 0$.

We shall call any minimizer of $\mathbb{I}$ an *H-harmonic function*, because when L is given by a Riemannian metric, the minimizers of $\mathbb{I}$ are just the harmonic functions, i.e., the constants. Finally we call

$$h := \min_{u \in H^{1,2}(M)} \mathbb{I}(u)$$

the harmonic value of L (or H).

2.1. THE CASE WHEN $H(x,p) = \frac{1}{2}|p - \theta_x|_x^2 + V(x)$. Suppose now that θ is a smooth 1-form and V is a smooth function on M and let

$$H(x,p) := \frac{1}{2}|p - \theta_x|_x^2 + V(x).$$

In this case the quasilinear elliptic equation (11) reduces to

$$d^*(du - \theta) = 0,$$

or equivalently

$$\Delta\, u = d^*\theta,$$

where Δ is the Laplacian of the Riemannian metric. In this simpler case all the results in Theorem 2.3 can be derived from well-known results about the Laplacian of a Riemannian manifold. Observe that when θ is closed a function u is H-harmonic iff $du - \theta$ is the unique harmonic 1-form in the class of θ.

2.2. GAS DYNAMICS ON A RIEMANNIAN MANIFOLD. Let $\rho\colon M \times \mathbb{R} \to \mathbb{R}$ be a C^∞ map. We shall say that ρ is *regular* if there exists a positive constant k such that for all $(x,t) \in M \times [0,\infty)$ we have

- $1/k < \frac{\partial}{\partial t}(t\,\rho(x,t)) < k$.

Given a regular ρ let us define $\tilde{\rho}$ as

$$\tilde{\rho}(x,t) := \int_0^t \rho(x,s)\, ds.$$

Endow M with a Riemannian metric and, given a closed 1-form ω, consider the Hamiltonian

$$H_\omega(x,p) := \frac{1}{2}\tilde{\rho}(x, |p - \omega_x|_x^2).$$

Since ρ is regular, it is easy to check that H_ω satisfies the controllable growth conditions. The quasilinear elliptic equation associated with H_ω is given by

$$d^*\left(\rho(\cdot\,, |du - \omega|^2)\,(du - \omega)\right) = 0.$$

This equation describes the stationary potential flow of a compressible fluid in a Riemannian manifold having prescribed periods determined by the cohomology class of ω. The function ρ is the density of the fluid. Locally we can write $\omega = d\phi$. Then $u(x) - \phi(x)$ is the velocity potential with velocity vector $\nabla(u - \phi)$.

In this setting, Theorem 2.3 is exactly the same as the Regular Theorem in the work of L. M. Sibner and R. J. Sibner [27] which motivated much of this section.

3. Weak KAM solutions and H-harmonic functions

In this section we use the results from the previous section to prove Theorem A and Corollary 1.

3.1. PROOF OF THEOREM A. We saw in the introduction that we always have $h \leq c$. Suppose first that $h = c$ and let u be a weak KAM solution. As we explained in the introduction, a weak KAM solution is Lipschitz and, by Rademacher's theorem, it is differentiable almost everywhere. At any point x of differentiability of u, we have

$$H(x, d_x u) = c.$$

On the other hand, it is well known (cf. [16, Section 7.3]) that Lipschitz functions are weakly differentiable and that $x \mapsto d_x u$ is bounded and hence they are contained in $H^{1,2}(M)$. Now observe that

$$\mathbb{I}(u) = \int_M H(x, d_x u)\, dx = c = h;$$

hence u is a minimizer of $\mathbb{I}$. By Theorem 2.3, u must be a smooth H-harmonic function. Since weak KAM solutions always exist and H-harmonic functions are unique up to constants, it follows that the set of weak KAM solutions coincides with the set of H-harmonic functions.

Suppose now that there exists a weak KAM solution u that is H-harmonic. Since u is a weak KAM solution $H(x, d_x u) = c$ for all $x \in M$, and since it is H-harmonic

$$\mathbb{I}(u) = \int_M H(x, d_x u)\, dx = c = h,$$

as desired. ∎

3.2. PROOF OF COROLLARY 1. The corollary is an immediate consequence of the following two lemmas.

LEMMA 3.1: *Given a Lagrangian L, there exists a function $\psi \in C^\infty(M, \mathbb{R})$ such that $h(L + \psi) = c(L + \psi) = 0$.*

Proof: Let H_L be the Hamiltonian associated with L and, given $\psi \in C^\infty(M, \mathbb{R})$, let $H_{L+\psi}$ be the Hamiltonian associated with $L + \psi$. We have $H_{L+\psi} = H_L - \psi$. Let u be an H_L-harmonic function and set $\psi(x) := H(x, d_x u)$. Then u is a solution of the Hamilton–Jacobi equation $H_{L+\psi}(x, d_x u) = 0$ and hence $c(L+\psi) =$

0. Now observe that the quasilinear elliptic equation of L coincides with the quasilinear elliptic equation of $L+\psi$. It follows that

$$h(L+\psi) = \int_M (H(x, d_x u) - \psi(x))\, dx = 0. \qquad \blacksquare$$

LEMMA 3.2: *Suppose that $h(L) = c(L)$. Then $h(L+\psi) = c(L+\psi)$ iff ψ is constant.*

Proof: It is obvious that if ψ is constant, then $h(L+\psi) = c(L+\psi)$. Let us show the converse. As in the previous lemma, note that the quasilinear elliptic equation of L coincides with the quasilinear elliptic equation of $L+\psi$ for any ψ. Let H_L be the Hamiltonian associated with L and let $H_{L+\psi}$ be the Hamiltonian associated with $L+\psi$. We have $H_{L+\psi} = H_L - \psi$. Let u be an H_L-harmonic function. Since $h(L) = c(L)$, Theorem A implies that $H(x, d_x u) = c(L)$ for all $x \in M$. Now observe that u is also $H_{L+\psi}$-harmonic; hence if $h(L+\psi) = c(L+\psi)$, by Theorem A we also have $H_{L+\psi}(x, d_x u) = c(L+\psi)$ for all $x \in M$. It follows that $c(L) - \psi(x) = c(L+\psi)$ for all $x \in M$ and hence ψ is constant. $\blacksquare$

In [19], Mañé introduced the concept of *generic property of a Lagrangian.* A property P is said to be *generic* for the Lagrangian L if there exists a generic set $\mathcal{O}$ (in the Baire sense) of the set $C^\infty(M, \mathbb{R})$ of all C^∞ functions from M to $\mathbb{R}$ such that, if $\psi \in \mathcal{O}$, the Lagrangian $L+\psi$ has the property P. One of Mañé's objectives in [19] was to show that Mather's theory of minimizing measures becomes much more accurate and stronger if one searches for generic properties. We now show:

PROPOSITION 3.3: *Given a Lagrangian L there exists a generic set $\mathcal{O} \subset C^\infty(M, \mathbb{R})$ such that if $\psi \in \mathcal{O}$, then $L+\psi$ has a unique (up to constants) positive weak KAM solution and a unique (up to constants) negative weak KAM solution and these solutions are* **not** *$H_{L+\psi}$-harmonic.*

Proof: Mañé proved in [19] that there exists a generic set $\mathcal{O}_1 \subset C^\infty(M, \mathbb{R})$ such that, if $\psi \in \mathcal{O}_1$, then $L+\psi$ has a unique minimizing measure and the restriction of the Euler-Lagrange flow to the support of this measure is uniquely ergodic and therefore transitive.

Fathi proved in [10] that if the Euler-Lagrange flow of a Lagrangian L is transitive on the Aubry-Mather set (i.e., the closure of the union of the supports of all the minimizing measures), then L has a unique (up to constants) positive weak KAM solution and a unique (up to constants) negative weak KAM solution. Now use Corollary 1 and Theorem A. $\blacksquare$

4. Schrödinger operators and minimal action functionals

Fix a Riemannian metric on M with volume one and consider a real C^∞ 1-form θ and a smooth function $V\colon M \to \mathbb{R}$. Let L be the convex and superlinear Lagrangian given by

$$L(x,v) := \frac{1}{2}|v|_x^2 + \theta_x(v) - V(x).$$

Its associated Hamiltonian is

$$H(x,p) = \frac{1}{2}|p-\theta_x|_x^2 + V(x).$$

The Dirac quantization rule says that to quantize H we have to replace p by the operator id, where d is the exterior differential. Let D be the operator from smooth $\mathbb{C}$-valued functions to smooth $\mathbb{C}$-valued 1-forms given by

$$Du = idu - u\theta.$$

We have the correspondence

$$\begin{aligned} &p \mapsto id, \\ &p - \theta \mapsto D, \\ &H \mapsto \mathbb{H}_{(\theta,V)} := \frac{D^*D}{2} + V, \end{aligned}$$

where D^* is the formal adjoint of D with respect to the L^2-inner product of functions and 1-forms. This means that for any smooth $\mathbb{C}$-valued function u and any smooth $\mathbb{C}$-valued 1-form η we have

$$\int_M \langle Du, \eta\rangle\, dx = \int_M u\overline{D^*\eta}\, dx,$$

and we consider in T^*M the Hermitian inner product induced by the Riemannian metric. But

$$\begin{aligned} \int_M \langle Du,\eta\rangle\, dx &= \int_M \langle idu - u\theta, \eta\rangle\, dx \\ &= \int_M u(i\overline{d^*\eta} - \langle\theta,\eta\rangle)\, dx \\ &= \int_M u\overline{(-id^*\eta - \langle\eta,\theta\rangle)}\, dx, \end{aligned}$$

hence

$$D^*\eta = -id^*\eta - \langle\eta,\theta\rangle.$$

Now we compute D^*D. We have

$$\begin{aligned} D^*Du &= D^*(idu - u\theta) \\ &= -id^*(idu - u\theta) - \langle idu - u\theta, \theta\rangle \\ &= d^*du + id^*(u\theta) - i\langle du, \theta\rangle + |\theta|^2 u. \end{aligned}$$

But

$$d^*(u\theta) = ud^*\theta - \langle du, \theta\rangle,$$

hence

$$D^*Du = d^*du - 2i\langle du, \theta\rangle + \left(id^*\theta + |\theta|^2\right) u.$$

If we recall that d^*d is the Laplacian Δ we obtain the following expression for the quantized Hamiltonian $\mathbb{H}_{(\theta,V)}$:

$$\mathbb{H}_{(\theta,V)}u = \frac{1}{2}\Delta u - i\langle du, \theta\rangle + \left(\frac{i}{2}d^*\theta + \frac{1}{2}|\theta|^2 + V\right) u.$$

We shall describe now the main properties of the operator $\mathbb{H}_{(\theta,V)}$. Our reference is Schigekawa's paper [26].

Let $L^2(M, \mathbb{C})$ be the set of all $\mathbb{C}$-valued square integrable functions with respect to the Riemannian measure. The operator $\mathbb{H}_{(\theta,V)}$ is known to be essentially self-adjoint as an operator in $L^2(M, \mathbb{C})$ and in the sequel we consider the smallest closed extension of $\mathbb{H}_{(\theta,V)}$, that we still denote by $\mathbb{H}_{(\theta,V)}$. The semigroup generated by $\mathbb{H}_{(\theta,V)}$ is a compact operator in $L^2(M, \mathbb{C})$ and hence $\mathbb{H}_{(\theta,V)}$ has a real discrete spectrum:

$$\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \to \infty.$$

4.1. PROPERTIES OF $\mathbb{H}_{(\theta,V)}$.

1. Let $H^1(M, \mathbb{Z}) \subset H^1(M, \mathbb{R})$ be the lattice given by the integral cohomology classes. Let ω be a closed 1-form. Then $[\omega]/2\pi \in H^1(M, \mathbb{Z})$ iff for any smooth closed curve γ

$$\int_\gamma \omega \equiv 0 \pmod{2\pi}.$$

Recall also that $[\omega]/2\pi \in H^1(M, \mathbb{Z})$ iff there exists a smooth function $\varphi: M \to S^1$ such that

$$\omega = \frac{d\varphi}{i\varphi} =: \alpha_\varphi.$$

Given such a φ, define a unitary operator $U_\varphi: L^2(M, \mathbb{C}) \to L^2(M, \mathbb{C})$ by

$$U_\varphi(u) = \varphi u.$$

Then we have

$$U_\varphi^* \mathbb{H}_{(\theta,V)} U_\varphi = \mathbb{H}_{(\theta+\alpha_\varphi,V)}.$$

2. For any u and v in $C^\infty(M,\mathbb{C})$ we have

$$\langle \mathbb{H}_{(\theta,V)} u, v\rangle_{L^2} = \frac{1}{2}\int_M \langle du + iu\theta, dv + iv\theta\rangle\, dx + \int_M V u\overline{v}\, dx.$$

3.

$$\begin{aligned}\lambda_0 &= \inf \operatorname{spec} \mathbb{H}_{(\theta,V)} \\ &= \inf_{\{u\in C^\infty(M,\mathbb{C}):\ |u|_{L^2}=1\}} \left(\frac{1}{2}\int_M |du + iu\theta|^2\, dx + \int_M V|u|^2\, dx\right).\end{aligned}$$

4. Suppose that $V \geq 0$. Then $\lambda_0 \geq 0$ with equality iff $V \equiv 0$, $d\theta = 0$ and $[\theta]/2\pi \in H^1(M,\mathbb{Z})$. Further, in this case, $\lambda_0 = 0$ is simple and the eigenfunction is $\overline{\varphi}$ if we write $\theta = \alpha_\varphi$, $\varphi \in C^\infty(M,S^1)$.
5. Let η be a smooth real valued 1-form. Given ε we consider the operator $\mathbb{H}_{(\theta+\varepsilon\eta,V)}$. The next result [26] describes the asymptotic behaviour of the least eigenvalue and its eigenfunction as $\varepsilon \to 0$. From the definition of $\mathbb{H}_{(\theta+\varepsilon\eta,V)}$, we have

$$\mathbb{H}_{(\theta+\varepsilon\eta,V)} = \mathbb{H}_{(\theta,V)} + \varepsilon H_1 + \varepsilon^2 H_2,$$

where

$$\begin{aligned}H_1 u &= -i\langle du,\eta\rangle + \left(\frac{i}{2}d^*\eta + \langle\theta,\eta\rangle\right) u,\\ H_2 u &= \frac{1}{2}|\eta|^2 u.\end{aligned}$$

THEOREM 4.1 ([26]): *Assume that the least eigenvalue λ_0 of $\mathbb{H}_{(\theta,V)}$ is simple. Let φ_0 be its eigenfunction such that $|\varphi_0|_{L^2} = 1$. Then for sufficiently small ε, the least eigenvalue $\lambda(\varepsilon)$ of $\mathbb{H}_{(\theta+\varepsilon\eta,V)}$ is simple. Moreover, choosing the eigenfunction $\varphi(\varepsilon)$ so that $\langle\varphi(\varepsilon),\varphi_0\rangle_{L^2} = 1$, $\lambda(\varepsilon)$ and $\varphi(\varepsilon)$ have asymptotic expansions*

$$\lambda(\varepsilon) = \lambda_0 + \varepsilon\nu_1 + \varepsilon^2\nu_2 + \cdots \quad \text{as } \varepsilon \to 0$$

and

$$\varphi(\varepsilon) = \varphi_0 + \varepsilon\varphi_1 + \varepsilon^2\varphi_2 + \cdots \quad \text{in } C^\infty(M,\mathbb{C}).$$

Here ν_i, φ_i, $i = 1,\ldots,$ are defined inductively as

$$\begin{aligned}\langle\varphi_i,\varphi_0\rangle_{L^2} &= 0,\\ \nu_i &= \langle H_1\varphi_{i-1},\varphi_0\rangle_{L^2} + \langle H_2\varphi_{i-2},\varphi_0\rangle_{L^2}\end{aligned}$$

and

$$(\mathbb{H}_{(\theta,V)} - \lambda_0)\varphi_i = -H_1\varphi_{i-1} - H_2\varphi_{i-2} + \sum_{k=1}^{k=i} \nu_k \varphi_{i-k}.$$

4.2. Proof of Theorem B. By property 1 in the previous subsection we have that the spectrum of $\mathbb{H}_{(\theta,V)}$ is the same as the spectrum of $\mathbb{H}_{(\theta-df,V)}$ for any smooth function $f\colon M \to \mathbb{R}$. Recall from Section 2 that a function f is H-harmonic iff

$$d^*(df - \theta) = 0.$$

By property 3 we have

$$\begin{aligned}\lambda_0 &= \inf \operatorname{spec} \mathbb{H}_{(\theta-df,V)} \\ &= \inf_{\{u\in C^\infty(M,\mathbb{C})\colon\, |u|_{L^2}=1\}} \left(\frac{1}{2}\int_M |du + iu(\theta - df)|^2\, dx + \int_M V|u|^2\, dx\right).\end{aligned}$$

If we take the function u which is identically equal to 1 and we use Theorem A, we have

$$\begin{aligned}\lambda_0 &\le \left(\frac{1}{2}\int_M |i(\theta - df)|^2\, dx + \int_M V\, dx\right) \\ &= \int_M H(x, d_x f)\, dx = h \le c,\end{aligned}$$

and the first inequality is an equality iff $u \equiv 1$ is an eigenfunction of $\mathbb{H}_{(\theta-df,V)}$ with eigenvalue λ_0. But the function $u \equiv 1$ is an eigenfunction of $\mathbb{H}_{(\theta-df,V)}$ with eigenvalue λ_0 iff for all $x \in M$,

$$\mathbb{H}_{(\theta-df,V)} u = \frac{1}{2}|\theta_x - d_x f|^2 + V(x) = \lambda_0.$$

Hence if $\lambda_0 = h$ we see using (4) that $h \ge c$. Since we know that we always have $h \le c$ we obtain $h = c$.

To complete the proof of Theorem B, we need to show that if $h = c$, then this common value is an eigenvalue of $\mathbb{H}_{(\theta,V)}$.

Let f be an H-harmonic function, i.e., f satisfies $d^*(df - \theta) = 0$. If we replace θ by $\theta - df$ in the Schrödinger operator we obtain

$$\mathbb{H}_{(\theta-df,V)} u = \frac{1}{2}\Delta u - i\langle du, \theta - df\rangle + H(df)u. \tag{17}$$

On account of Theorem A, if $h = c$ then the H-harmonic function f is also a weak KAM solution. This implies that for all $x \in M$

$$H(x, d_x f) = \frac{1}{2}|d_x f - \theta_x|^2 + V(x) = c.$$

If we use this relation in (17) we obtain

$$\mathbb{H}_{(\theta-df,V)}u = \frac{1}{2}\Delta u - i\langle du, \theta - df\rangle + cu.$$

Hence the function $u \equiv 1$ is an eigenfunction of $\mathbb{H}_{(\theta-df,V)}$ with eigenvalue c, thus concluding the proof of Theorem B. ∎

4.3. An example showing that when $h = c$, this common value may not be the smallest eigenvalue. Suppose that $V \equiv 0$. Let L_r be the Lagrangian

$$L_r(x, v) = \frac{1}{2}|v|_x^2 + r\,\theta_x(v),$$

where r is a positive real number. Let H_r be the Hamiltonian associated with L_r. Using equality (4) we have

$$\begin{aligned} c_r := c(L_r) &= \inf_{u\in C^\infty(M,\mathbb{R})} \max_{x\in M} H_r(x, d_x u) \\ &= \inf_{u\in C^\infty(M,\mathbb{R})} \max_{x\in M} \frac{1}{2}|d_x u - r\,\theta_x|^2. \end{aligned}$$

Hence

$$c_r = r^2\, c_1.$$

Suppose now that we choose a Riemannian metric and a 1-form θ on a manifold M with the following properties:

1. θ is closed and $0 \neq [\theta]/2\pi \in H^1(M, \mathbb{Z})$.
2. θ is harmonic and has constant norm with respect to the Riemannian metric.

It is quite simple to give examples satisfying properties 1 and 2. Let M be a closed manifold with non-negative Ricci curvature and with $H^1(M, \mathbb{Z}) \neq 0$. Let $q \in H^1(M, \mathbb{Z})$ be a non-zero integral class. Let θ be the unique harmonic representative in the class $q/2\pi$. On every manifold of non-negative Ricci curvature the harmonic forms are parallel, hence θ has constant norm. Another example of a form θ that satisfies properties 1 and 2 but is *not* parallel is given by the torus of revolution obtained by rotating a round circle around an axis in $\mathbb{R}^3$. The meridians of the surface foliate the torus and give rise to a vector field on the surface with unit norm. The 1-form dual to this vector field is harmonic, has constant norm, but is not parallel.

Observe now that since θ has constant norm, the function $u \equiv 1$ is a weak KAM solution of L_r for all r. Moreover, $u \equiv 1$ is also H_r-harmonic since $d^*\theta = 0$. It follows that $c_r = h(L_r)$ for all r and that c_1 is precisely $1/2|\theta_x|^2$ for all $x \in M$.

By Theorem B, c_r is an eigenvalue of $\mathbb{H}_{(r\theta,0)}$. Now take $r = n$, where n is a positive integer. Then $n[\theta]/2\pi \in H^1(M, \mathbb{Z})$ and therefore the smallest eigenvalue of $\mathbb{H}_{(n\theta,0)}$ is $\lambda_0 = 0$ but $c_n \to \infty$ as $n \to \infty$.

4.4. EXPANSION OF $\lambda_0(\varepsilon)$. Let ε be a small parameter. Since the smallest eigenvalue of $\mathbb{H}_{(0,0)} = \Delta/2$ is simple, then $\lambda_0(\varepsilon)$, the smallest eigenvalue of $\mathbb{H}_{(\varepsilon\theta,0)}$, is also simple and the map $\varepsilon \mapsto \lambda_0(\varepsilon)$ is real analytic by standard perturbation theory [17]. We will use the expansion given in Subsection 4.1, property 5 to show the following:

PROPOSITION 4.2: *For ε small we have*

$$\lambda_0(\varepsilon) = \varepsilon^2 h + \text{ higher order terms,}$$

and

$$\lambda_0(\varepsilon) = \varepsilon^2 h,$$

for all ε sufficiently small iff $h = c$.

Proof: We can assume without loss of generality that $d^*\theta = 0$, since otherwise we replace θ by $\theta - df$, where f is an H-harmonic function (recall that $\mathbb{H}_{(\varepsilon\theta,0)}$ and $\mathbb{H}_{(\varepsilon(\theta-df),0)}$ have the same spectrum). We now use property 5 of Subsection 4.1. We need to compute ν_1 and ν_2 in Theorem 4.1. Note that

$$\begin{aligned} H_1 u &= - i\langle du, \theta\rangle, \\ H_2 u &= \frac{1}{2}|\theta|^2 u. \end{aligned}$$

Since $\lambda_0(0) = 0$ and $\varphi_0 \equiv 1$ we have

$$\nu_1 = \langle H_1\varphi_0, \varphi_0\rangle = 0,$$

and

$$\begin{aligned} \nu_2 &= \langle H_1\varphi_1, \varphi_0\rangle + \langle H_2\varphi_0, \varphi_0\rangle \\ &= \langle H_1\varphi_1, \varphi_0\rangle + \frac{1}{2}\int_M |\theta|_x^2\, dx \\ &= \langle H_1\varphi_1, \varphi_0\rangle + h. \end{aligned}$$

The function φ_1 is determined by the equation

$$\mathbb{H}_{(0,0)}\varphi_1 = \frac{1}{2}\Delta\varphi_1 = -H_1\varphi_0 + \nu_1\varphi_0 = 0.$$

Hence φ_1 must be constant and $\langle \varphi_1, \varphi_0\rangle = 0$, which implies $\varphi_1 \equiv 0$. Therefore $\nu_2 = h$ as desired.

We now compute the coefficients ν_i for $i \geq 3$:

$$\begin{aligned}\nu_i &= \langle H_1\varphi_{i-1}, \varphi_0\rangle + \langle H_2\varphi_{i-2}, \varphi_0\rangle \\ &= -i\int_M \langle d\varphi_{i-1}, \theta\rangle\, dx + \langle H_2\varphi_{i-2}, \varphi_0\rangle \\ &= -i\int_M \varphi_{i-1}\overline{d^*\theta}\, dx + \langle H_2\varphi_{i-2}, \varphi_0\rangle \\ &= \langle H_2\varphi_{i-2}, \varphi_0\rangle,\end{aligned}$$

since $d^*\theta = 0$.

Suppose that $h = c$. In this case, by Theorem A, the constant functions are weak KAM solutions and hence for all $x \in M$ we must have

$$\frac{1}{2}|\theta_x|^2 = h = c.$$

It follows that

$$\begin{aligned}\nu_i &= \langle H_2\varphi_{i-2}, \varphi_0\rangle \\ &= c\langle \varphi_{i-2}, \varphi_0\rangle \\ &= 0.\end{aligned}$$

Therefore, for all ε sufficiently small $\lambda_0(\varepsilon) = \varepsilon^2 h$.

Conversely, suppose that for some ε sufficiently small $\lambda_0(\varepsilon) = \varepsilon^2 h$. By Theorem B, $\varepsilon^2 h = \varepsilon^2 c$ and thus $h = c$. ∎

5. Schrödinger norm, L^2-norm and the stable norm

Suppose that L is just given by a Riemannian metric, that is,

$$L(x, v) = \frac{1}{2}|v|_x^2.$$

In this case it is known that Mather's α function and the stable norm in cohomology ([23]) are related by

$$\|[\omega]\|_s = \sqrt{2\alpha([\omega])} = \sqrt{2\,c(L-\omega)}. \tag{18}$$

Let ω be a closed 1-form. By considering the Lagrangian $L - \omega$, whose Euler-Lagrange flow coincides with the geodesic flow, we have functions on $H^1(M, \mathbb{R})$ given by

$$\begin{aligned}&[\omega] \mapsto h(L-\omega), \\ &[\omega] \mapsto \lambda_0(\omega) = \text{smallest eigenvalue of } \mathbb{H}_{(\omega,0)}.\end{aligned}$$

The Hamiltonian associated to $L - \omega$ is given by

$$H(x,p) = \frac{1}{2}|p + \omega_x|_x^2.$$

By definition, the harmonic value of $L - \omega$ can be written as

$$h(L-\omega) = \inf_{u \in C^\infty(M,\mathbb{R})} \int_M \frac{1}{2}|d_x u + \omega_x|_x^2\, dx.$$

The infimum is achieved exactly when $du + \omega$ is the unique harmonic form in the class $[\omega]$. Hence, $\sqrt{2h(L-\omega)}$ is nothing but the L^2-norm of the unique harmonic 1-form in the class $[\omega]$. We denote it by $|[\omega]|_{L^2}$.

By standard perturbation theory and Proposition 4.2 the function

$$[\omega] \mapsto \lambda_0(\omega)$$

is real analytic and strictly convex in a neighborhood of the origin. Hence for σ small and positive, $\lambda_0^{-1}(\sigma)$ is a convex hypersurface in $H^1(M,\mathbb{R})$ symmetric about the orgin. As we explained in the introduction, this hypersurface can be used to define a third norm in $H^1(M,\mathbb{R})$. We call it the *Schrödinger norm* and it is given by

$$|[\omega]|_{Schr,\sigma} = \frac{\sqrt{2\sigma}}{r},$$

where r is the unique positive real number such that $r[\omega] \in \lambda_0^{-1}(\sigma)$.

5.1. Proof of Theorem C. Without loss of generality we shall assume that ω is harmonic. Since $\lambda_0(r[\omega]) = \sigma$, Theorem B implies that

$$\sigma \leq h(L - r\omega) \leq c(L - r\omega). \tag{19}$$

Equivalently,

$$\sigma \leq r^2 h(L-\omega) \leq r^2 c(L-\omega),$$

which in turn implies

$$\frac{2\sigma}{r^2} \leq 2h(L-\omega) \leq 2c(L-\omega).$$

Using (18) and that $|[\omega]|_{L^2} = \sqrt{2h(L-\omega)}$ we obtain

$$|[\omega]|_{Schr,\sigma} \leq |[\omega]|_{L^2} \leq |[\omega]|_s.$$

Suppose that $|[\omega]|_{Schr,\sigma} = |[\omega]|_{L^2}$. This means that in (19) the first inequality is an equality. By Theorem B this implies that $h(L - r\omega) = c(L - r\omega)$ and, by

Theorem A, the constant functions are weak KAM solutions of $L - r\omega$. This implies that

$$x \mapsto \frac{1}{2}|r\omega_x|^2$$

is constant, i.e., the norm $|\omega_x|$ is the same for all x. Suppose that $\|[\omega]\|_{L^2} = \|[\omega]\|_s$. This means that in (19) the second inequality is an equality and, as before, the norm $|\omega_x|$ is the same for all x. Finally, suppose that the norm $|\omega_x|$ is the same for all x. Hence the constant functions are weak KAM solutions of $L-\omega$ and, by Theorem A, $h(L-\omega) = c(L-\omega)$. It follows that $\|[\omega]\|_{L^2} = \|[\omega]\|_s$. By Proposition 4.2, $\sigma = r^2 h(L-\omega)$ and thus $\|[\omega]\|_{Schr,\sigma} = \|[\omega]\|_{L^2}$.

Finally, observe that Proposition 4.2 and the definition of the Schrödinger norm imply that

$$\lim_{\sigma \to 0} \|[\omega]\|_{Schr,\sigma} = \|[\omega]\|_{L^2}. \qquad \blacksquare$$

5.2. Proof of Corollary 2. Suppose that for some σ the three norms coincide and let

$$\{\omega^1, \dots, \omega^n\}$$

be a basis of the space of harmonic 1-forms and such that the cohomology class of each ω^i is integral. By hypothesis n is also the dimension of M. By Theorem C each harmonic form has constant norm and hence, for each $x \in M$,

$$\{\omega_x^1, \dots, \omega_x^n\}$$

is a basis of T_x^*M. Fix a point $x_0 \in M$ and let us consider the Abel-Jacobi map $A\colon M \to \mathbb{T}^n$ defined as follows, where $\mathbb{T}^n$ is the n-torus. Given $x \in M$, let γ be a smooth path connecting x_0 to x. Set

$$A(x) = \left(\int_\gamma \omega^1 (\operatorname{mod} \mathbb{Z}), \dots, \int_\gamma \omega^n (\operatorname{mod} \mathbb{Z})\right).$$

Given $v \in T_xM$ an easy computation shows that

$$d_x A(v) = (\omega_x^1(v), \dots, \omega_x^n(v)),$$

which implies that A is a local diffeomorphism and therefore a finite covering map. It follows that M is diffeomorphic to the n-torus.

Let us prove that M is free of conjugate points. Given a covector $p \in T_x^*M$, there exists a unique harmonic 1-form ω such that $\omega_x = p$. By the Hamilton-Jacobi theorem, the graph of ω, G_ω, is a Lagrangian submanifold invariant under the cogeodesic flow ϕ_t^*. It follows that $T^*M \ni (x,p) \mapsto T_{(x,p)}G_\omega$ is a continuous

Lagrangian distribution invariant under ϕ_t^*. By a theorem of R. Mañé [20], this implies that M is free of conjugate points. By Burago and Ivanov's proof of the Hopf conjecture [3] M is flat.

Next observe that if M is a flat torus, every harmonic form is parallel and hence has constant Riemannian norm. By Theorem C the three norms coincide for all σ.

To complete the proof of the corollary suppose that $|\cdot|_{Schr,\sigma}$ is independent of σ. By Theorem C we know that

$$\lim_{\sigma\to 0} |[\omega]|_{Schr,\sigma} = |[\omega]|_{L^2},$$

and hence $|[\omega]|_{Schr,\sigma}$ coincides with the L^2-norm for all σ. Again, by Theorem C the three norms coincide for all σ. ∎

5.3. Other consequences. It seems interesting to relate the results that we obtained so far with the results obtained in [25].

Let M be a closed manifold endowed with a C^∞ Riemannian metric. Given x and y in M, let $\gamma_{x,y}$: $[0,\ell(\gamma_{x,y})] \to M$ be a unit speed geodesic arc joining x to y with length $\ell(\gamma_{x,y})$. Given $T>0$, the set of all $\gamma_{x,y}$ with $\ell(\gamma_{x,y}) \leq T$ is finite and its cardinality is locally constant for an open full measure subset of $M\times M$.

Given a closed 1-form ω, let $P(\omega)$ be the topological pressure of ω, where we think of ω as a function ω: $TM \to \mathbb{R}$. In [25] we showed that

$$P(\omega) = \lim_{T\to\infty} \frac{1}{T} \log \int_{M\times M} \left(\sum_{\{\gamma_{x,y}:\, \ell(\gamma_{x,y})\leq T\}} e^{\int_{\gamma_{x,y}} \omega} \right) dxdy. \tag{20}$$

We also observed in [25] that

$$|[\omega]|_s \leq P(\omega) \leq h_{top} + |[\omega]|_s, \tag{21}$$

where h_{top} is the topological entropy of the geodesic flow.

Observe that from Theorem B, (20) and (21) we obtain right away that

$$\sqrt{2\,\lambda_0(\omega)} \leq \lim_{T\to\infty} \frac{1}{T} \log \int_{M\times M} \left(\sum_{\{\gamma_{x,y}:\, \ell(\gamma_{x,y})\leq T\}} e^{\int_{\gamma_{x,y}} \omega} \right) dxdy.$$

It is suggestive to compare this inequality (which is interesting only for ω in a neighborhood of the origin) with Schigekawa's observation in [26, Equation 4.9]. He observes that if $\lambda_0(\omega)$ is simple then

$$\lambda_0(\omega) = \lim_{T\to\infty} \frac{1}{T} \log \int_M e^{i\int_0^T \omega(X(s,x))\circ dX(s,x)}\, \delta_y(X(T,x))\, dy,$$

where $\{X(s,x);\ s \geq 0\}$ is a Brownian motion on M starting at x,

$$\int_0^T \omega(X(s,x)) \circ dX(s,x)$$

is the stochastic line integral of the 1-form ω along the paths $\{X(t,x)\}_{t\geq 0}$ and

$$e^{i\int_0^T \omega(X(s,x))\circ dX(s,x)}\,\delta_y(X(T,x))$$

is a pairing of a smooth Wiener functional $e^{i\int_0^T \omega(X(s,x))\circ dX(s,x)}$ and a generalized Wiener functional $\delta_y(X(T,x))$, where δ_y is the Dirac measure at y.

5.4. QUESTIONS. The previous results raise a couple of attractive questions:

1. Suppose that for some σ, the norm $|\cdot|_{Schr,\sigma}$ is Euclidean, i.e., it derives from an inner product in $H^1(M,\mathbb{R})$. What can be said about the Riemannian metric on M?
2. The unit sphere of the Schrödinger norm is a real analytic convex hypersurface of $H^1(M,\mathbb{R})$. How much does the geometry of this hypersurface reflect the geometry of M?

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